A Convergent Finite Volume type O-method on Evolving Surfaces

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Abstract. We present a finite volume scheme for anisotropic diffusion on evolving hypersurfaces. The underlying motion is assumed to be described by a fixed, not necessarily normal, velocity field. The ingredients of the numerical method are an approximation of the family of surfaces by a family of interpolating polygonal meshes, where grid vertices move on motion trajectories, a consistent finite volume discretization of the induced transport on the cells (polygonal patches), and a proper incorporation of a diffusive flux balance at polygonal faces. The main stability results and convergence estimate are obtained.

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INTRODUCTION

In many applications, evolution problems do not reside on a flat Euclidean domain but on a curved hypersurface. Frequently this surface is itself evolving in time driven by some velocity field. In [1] Dziuk and Elliot proposed a finite element scheme for the numerical simulation of diffusion processes on such evolving surfaces, and in [2] a finite volume variant was proposed for simulation on simplicial meshes. In this paper we introduce a finite volume methodology for simulation of diffusion process on general evolving polygonal meshes. our finite volume is closely related to the pioneer work by C. Le Potier in [3] and the work of K. Lipnikov, M. Shashkov and I. Yotov in [4].

MATHEMATICAL MODEL

We consider a family of compact, smooth, and oriented hypersurfaces $\Gamma(t) \subset \mathbb{R}^n$ ($n = 2$, 3) for $t \in [0,t_{max}]$ generated by a time dependent function $\Phi : [0,t_{max}] \times \Gamma_0 \to \mathbb{R}^n$ defined on a reference surface Γ_0 with $\Phi(t,\Gamma_0) = \Gamma(t)$. Let us assume that Γ_0 is C^3 smooth and that $\Phi \in C^1([0,t_{max}], C^3(\Gamma_0))$. For simplicity we assume the reference surface Γ_0 to coincide with $\Gamma(0)$ (cf. 2).

We denote by $v = \partial_t\Phi$ the velocity of material points. The evolution of a conservative material quantity u with $u(t,\cdot) : \Gamma(t) \to \mathbb{R}$, which is propagated with the surface and simultaneously undergoes a linear diffusion on the surface, is governed by the parabolic equation

$$\dot{u} + u\nabla_\Gamma \cdot v - \nabla_\Gamma \cdot (\mathscr{D}\nabla_\Gamma u) = g \quad \text{on } \Gamma = \Gamma(t), \tag{1}$$

where $\dot{u} = \frac{\mathrm{d}}{\mathrm{d}t}u(t,x(t))$ is the (advective) material derivative of u, $\nabla_\Gamma \cdot v$ the surface divergence of the vector field v, $\nabla_\Gamma u$ the surface gradient of the scalar field u, g a source term with $g(t,\cdot) : \Gamma(t) \to \mathbb{R}$ and $\mathscr{D}$ a diffusion tensor on the tangent bundle. Here we assume a symmetric, uniformly coercive C^2 diffusion tensor field on whole $\mathbb{R}^n$ to be given, whose restriction on the tangent plane is then effectively incorporated in the model. With a slight misuse of notation, we denote this global tensor field also by $\mathscr{D}$. Furthermore, we impose an initial condition $u(0,\cdot) = u_0$ at time 0, and treat the case of surfaces with boundary. We then consider a Dirichlet boundary condition $u(t,\cdot) := u|_{\partial\Gamma}(t,\cdot)$ on the boundary $\partial\Gamma(t)$. Let us assume that the mappings $(t,x) \to u(t,\Phi(t,x))$, $v(t,\Phi(t,x))$, and $g(t,\Phi(t,x))$ are $C^1([0,t_{max}], C^3(\Gamma_0))$, $C^0([0,t_{max}], C^3(\Gamma_0))$ and $C^1([0,t_{max}], C^1(\Gamma_0))$ regular, respectively.

DERIVATION OF THE FINITE VOLUME SCHEME

For the ease of presentation we restrict ourselves to the case of two dimensional surfaces in $\mathbb{R}^3$. let us give some preliminary definitions

Definition 0.1 *(Cell, cell center and vertices) We call cell S a continuous 3D fan of triangles, where each triangle shares an edge with the preceding triangle, and all triangles share a common pivot point X_S called cell center or center point. The corner points $p_i,\ (i = 0,1,\cdots)$ as depicted on Figure 1 (left) will be called vertices.*

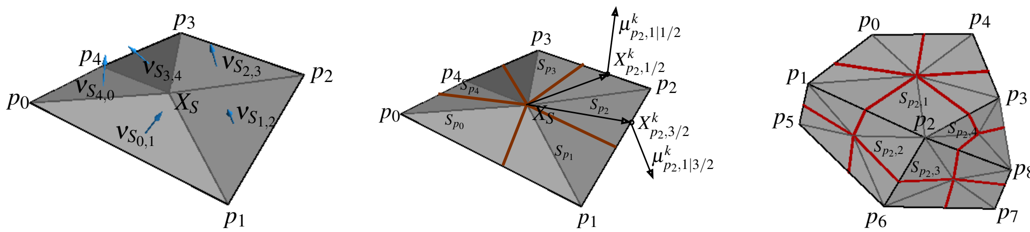

Figure 1. Admissible cell S and corresponding normals $\nu_{S_{i,j}}$ to triangles (left), sub-cells S_{p_i} of S, virtual unknowns $X^k_{p_2,1/2}$, $X^k_{p_2,3/2}$ of $S_{p_2} \equiv S_{p_2,1}$ and associated contravariant vectors $\mu^k_{p_2,1/2}$, $\mu^k_{p_2,3/2}$ (middle), sub-cells $S_{p_2,j}$ around the vertex p_2 (right).

Definition 0.2 *(Admissible cell) Let S be a cell, X_S its center point, and p_i $(i = 0,1,\cdots,n_S - 1)$ its n_S vertices. For a given vertex p_i we denote by $\nu_{S_{i,j}} = \overrightarrow{X_S p_i} \wedge \overrightarrow{X_S p_j} / (\| \overrightarrow{X_S p_i} \wedge \overrightarrow{X_S p_j} \|)$ $(j \equiv (i+1) \mod n_S)$ the normal of the triangle $[X_S, p_i, p_j]$ if it has a non zero measure. We will then call S admissible cell if for any i, $j \equiv (i+1) \mod n_S$, $l \equiv (j+1) \mod n_S$, $m \in \{0,1,\cdots,n_S - 1\}$ $\|\overrightarrow{X_S p_i}\| \le \max_{l,m} \|\overrightarrow{p_l p_m}\|$ and $\nu_{S_{i,j}} \cdot \nu_{S_{j,l}} > 0$ for well defined normals.*

Definition 0.3 *(admissible polygonal surface) We define an admissible polygonal surface Γ_h as a C^0 union of admissible cells where for two different cells S_i, $S_k \subset \Gamma_h$, $S_i \cap S_k$ is either an interface $\sigma = S_i | S_k = [p_i p_j]$, a vertex p_i, or is merely empty.*

We now consider a sequence of admissible polygonal surfaces $\{\Gamma^k_h\}_{k=0,\ldots k_{max}}$ with Γ^k_h interpolating $\Gamma(t_k)$ for $t_k = k\tau$ and $k_{max}\tau = t_{max}$. Here, h indicates the maximal diameter of a cell on the whole sequence of polygonization, τ the time step size and k the index of a time step. All polygonisation share the same grid topology, and given the set of vertices p^0_j

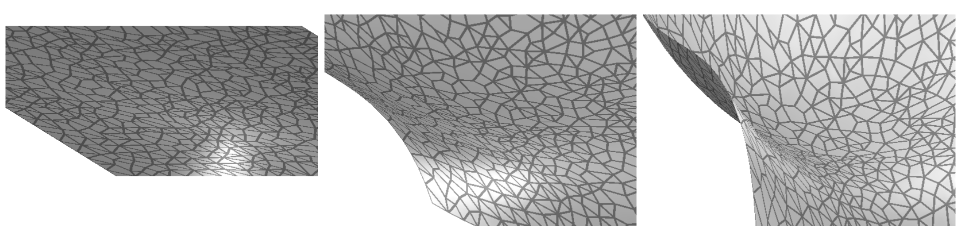

Figure 2. Zoom of sequence of polygonization Γ^k_h interpolating a surface in its evolution.

on the initial polygonal surface Γ^0_h, the vertices of Γ^k_h lie on motion trajectories. Thus, they are evaluated based on the flux function Φ, i. e. $p_j(t_k) = \Phi(t_k, p^0_j)$. Upper indices denote the explicit geometric realization at the corresponding time step. We further assume that at each time step t_k, the Euclidean distance from any center point X^k_S to $\Gamma(t_k)$ is less than h^2 and the sub-cells $S^k_{i,j} := [X^k_S, p^k_i, p^k_{i+1}]$ ($i+1$ being the following index) are uniformly regular triangles. At each time step t_k, we consider a virtual subdivision of each cell S^k with n_S vertices into n_S polygonal sub-cells $\{S_{p_i}\}_{i=1,\cdots,n_S}$ sharing X_S as depicted on Figure 1 (middle). This induces a subdivision of each edge $\sigma = [p_i, p_j] \subset \partial S$ into two sub-edges. Let us consider a vertex p_i and reorganize its surrounding cells S_j counter clockwise (*cf.* Figure 1 (right)). We call $\sigma_{p_i,j-1/2}$ and $\sigma_{p_i,j+1/2}$ the two sub-edges of S_j incident at p_i, and on each of these sub-edges, we respectively put the virtual unknowns $X^k_{p_i,j-1/2}$ and $X^k_{p_i,j+1/2}$. We recall that the index "$j-1/2$" refers to the next sub-edge.
On $S_{p_i,j}$, sub-cell of S_j containing p_i, we define the covariant vectors $e^k_{p_i,j|j-1} := X^k_{p_i,j-1/2} - X^k_{S_j}$ and $e^k_{p_i,j|j+1} := X^k_{p_i,j+1/2} - X^k_{S_j}$ and their contravariant counter part $\mu^k_{p_i,j|j-1}$ and $\mu^k_{p_i,j|j+1}$ in $T^k_{p_i} := Span\{e^k_{p_i,j|j-1}, e^k_{p_i,j|j+1}\}$ such that

$\mu^k_{p_i,j|j-1}\cdot e^k_{p_i,j|j-1}=1$, $\mu^k_{p_i,j|j-1}\cdot e^k_{p_i,j|j+1}=0$, $\mu^k_{p_i,j|j+1}\cdot e^k_{p_i,j|j-1}=0$ and $\mu^k_{p_i,j|j+1}\cdot e^k_{p_i,j|j+1}=1$ (*cf.* Figure 1 (middle)). Using this dual system of vectors, we define on $S_{p_i,j}$ the natural approximation of tangential gradient $\nabla u(t_k,\cdot)|_{S_{p_i,j}}\approx\nabla^k_{p_i,j}u:=[u(\mathscr{P}^k(X^k_{p_i,j-1/2}))-u(\mathscr{P}^k(X^k_{S_j}))]\mu^k_{p_i,j|j-1}+[u(\mathscr{P}^k(X^k_{p_i,j+1/2}))-u(\mathscr{P}^k(X^k_{S_j}))]\mu^k_{p_i,j|j+1}$. Here, $\mathscr{P}^k$ is the orthogonal projection onto the smooth surface when applied to interior points of Γ^k, and the orthogonal projection onto the boundary of the smooth surface when applied to boundary points of Γ^k. Based on these notational preliminaries, we can now derive a suitable finite volume discretization. Let us integrate (1) in $\{(t,x)|t\in[t_k,t_{k+1}],x\in S^{l,k}(t)\}$ ($S^{l,k}(t):=\Phi(t,\mathscr{P}^k(S^k)\cap\Gamma(t_k))$ where $\mathscr{P}^k$ is taken to be the projection of onto a suitable extension of $\Gamma(t_k)$.) We should mention here that $\mathscr{P}^k$ is context depend.

$$\int_{t_k}^{t_{k+1}}\int_{S^{l,k}(t)}g\,da\,dt\approx\tau m^{k+1}_S G^{k+1}_S \tag{2}$$

where $G^{k+1}_S=g\left(t_{k+1},\mathscr{P}^{k+1}X^{k+1}_S\right)$ and m^{k+1}_S the measure of S^{k+1}. The use of the Leibniz formula leads to the following approximation of the material derivative

$$\int_{t_k}^{t_{k+1}}\int_{S^{l,k}(t)}(\dot{u}+u\nabla_\Gamma\cdot v)\,da\,dt \quad = \quad \int_{S^{l,k}(t_{k+1})}u\,da-\int_{S^{l,k}(t_k)}u\,da\approx m^{k+1}_S u\left(t_{k+1},\mathscr{P}^{k+1}X^{k+1}_S\right)-m^k_S u\left(t_k,\mathscr{P}^k X^k_S\right). \tag{3}$$

Furthermore, integrating the elliptic term again over the temporal evolution of a lifted cell and applying Gauss' theorem, we derive the following approximation:

$$\begin{aligned}\int_{t_k}^{t_{k+1}}\int_{S^{l,k}(t)}\nabla_\Gamma\cdot(\mathscr{D}\nabla_\Gamma u)\,da\,dt &= \int_{t_k}^{t_{k+1}}\int_{\partial S^{l,k}(t)}\mathscr{D}\nabla_\Gamma u\cdot\mu_{\partial S^{l,k}(t)}\,dl\,dt\\ \approx\quad \tau\sum_{p_i}&\left(m^{k+1}_{p_i,j(S)+1/2}(\mathscr{D}^{k+1}_S\nabla^{k+1}_{p_i,j(S)}u)\cdot n^{k+1}_{p_i,j(S)|j(S)+1}+m^{k+1}_{p_i,j(S)-1/2}(\mathscr{D}^{k+1}_S\nabla^{k+1}_{p_i,j(S)}u)\cdot n^{k+1}_{p_i,j(S)|j(S)-1}\right)\end{aligned} \tag{4}$$

where $\mu_{\partial S^{l,k}(t)}$ is the unit outward co-normal on $\partial S^{l,k}(t)$ tangential to $\Gamma(t)$ and $m^{k+1}_{p_i,j(S)+1/2}$ the measure of $\sigma^{k+1}_{p_i,j(S)+1/2}$. The summation is done over the vertices p_i of the topological cell S, $j(S)$ denotes the reordered index number of S as a cell of the cluster of cells around p_i; $\mathscr{D}^{k+1}_S:=\mathscr{D}(t_{k+1},X^{k+1}_S)$, and $n^{k+1}_{p_i,j(S)|j(S)+1}$ respectively $n^{k+1}_{p_i,j(S)|j(S)+1}$ being defined as the unit outward co-normal vectors of $S^k_{p_i,j(S)}$. These last vectors belong to the plane $T^{k+1}_{p_i,j}$ and are respectively normal to $\sigma^{k+1}_{p_i,j(S)+1/2}$ and $\sigma^{k+1}_{p_i,j(S)-1/2}$. To close our equation, we impose a flux continuity on sub-edges which is numerically expressed as

$$\begin{aligned}&m^{k+1}_{p_i,j+1/2}\left[\left(U^{k+1}_{p_i,j+1/2}-U^{k+1}_{p_i,j}\right)\lambda^{k+1}_{p_i,j|j+1}+\left(U^{k+1}_{p_i,j-1/2}-U^{k+1}_{p_i,j}\right)\lambda^{k+1}_{p_i,j-1|j|j+1}\right]\\ +\quad &m^{k+1}_{p_i,j+1/2}\left[\left(U^{k+1}_{p_i,j+3/2}-U^{k+1}_{p_i,j+1}\right)\lambda^{k+1}_{p_i,j+2|j+1|j}+\left(U^{k+1}_{p_i,j+1/2}-U^{k+1}_{p_i,j+1}\right)\lambda^{k+1}_{p_i,j+1|j}\right]=0.\end{aligned} \tag{5}$$

where $\lambda^{k+1}_{p_i,j|j+1}:=\mathscr{D}^{k+1}_S\mu^{k+1}_{p_i,j|j+1}\cdot n^{k+1}_{p_i,j|j+1}$, $\lambda^{k+1}_{p_i,j-1|j|j+1}:=\mathscr{D}^{k+1}_S\mu^{k+1}_{p_i,j|j-1}\cdot n^{k+1}_{p_i,j|j+1}$, and $\lambda^{k+1}_{p_i,j+1|j|j-1}$, $\lambda^{k+1}_{p_i,j|j-1}$ similarly defined. We denote by $U^{k+1}_{p_i,j}$ the cell center unknowns and $U^{k+1}_{p_i,j+1/2}$ the sub-edge unknowns. (5) gives a local relations $M^{k+1}_{p_i}\bar{U}^{k+1}_{p_i,j+1/2}=N^{k+1}_{p_i}\bar{U}^{k+1}_{p_i,j}$ between the vector of cell centers unknowns $\bar{U}^{k+1}_{p_i,j}$, and the vector of sub-edge unknowns $\bar{U}^{k+1}_{p_i,j+1/2}$ around vertices p_i. The behavior of the system depends on the matrices $M^{k+1}_{p_i}$, $N^{k+1}_{p_i}$ and

$$Q^{k+1}_{p_i,j}:=\begin{pmatrix} m^{k+1}_{p_i,j+1/2}\lambda^{k+1}_{p_i,j|j+1} & m^{k+1}_{p_i,j-1/2}\lambda^{k+1}_{p_i,j+1|j|j-1}\\ m^{k+1}_{p_i,j+1/2}\lambda^{k+1}_{p_i,j-1|j|j+1} & m^{k+1}_{p_i,j-1/2}\lambda^{k+1}_{p_i,j|j-1}\end{pmatrix} \tag{6}$$

We then assume from now on that the sub-edge points are chosen such that $Q^{k+1}_{p_i,j}$ is positive definite and $(M^{k+1}_{p_i})^{-1}N^{k+1}_{p_i}$ is positive. It is then clear that an admissible polygonization must allow this set-up. It is worth mentioning here That if all incident angles at vertices are less than 180 degrees, a good choice of center point X^k_S of each cell S^k combined with a suitable optimization procedure around the vertices will always guarantee the above mentioned condition. The scheme described in [2] is then a good example and $Q^{k+1}_{p_i,j}$ is a diagonal matrix in that case. We recall (2), (3), and (4) to obtain the following equation on each cell

$$\begin{aligned}&m^{k+1}_S U^{k+1}_S-m^k_S U^k_S-\tau\sum_{p_i}m^{k+1}_{p_i,j(S)+1/2}\left[\left(U^{k+1}_{p_i,j(S)+1/2}-U^{k+1}_{p_i,j(S)}\right)\lambda^{k+1}_{p_i,j(S)|j(S)+1}+\left(U^{k+1}_{p_i,j(S)-1/2}-U^{k+1}_{p_i,j(S)}\right)\lambda^{k+1}_{p_i,j(S)-1|j(S)|j(S)+1}\right]\\ +\quad &m^{k+1}_{p_i,j(S)-1/2}\left[\left(U^{k+1}_{p_i,j(S)+1/2}-U^{k+1}_{p_i,j(S)}\right)\lambda^{k+1}_{p_i,j(S)+1|j(S)|j(S)-1}+\left(U^{k+1}_{p_i,j(S)-1/2}-U^{k+1}_{p_i,j(S)}\right)\lambda^{k+1}_{p_i,j(S)|j(S)-1}\right]=\tau m^{k+1}_S G^{k+1}_S\end{aligned} \tag{7}$$

which together with (5), the initial condition and the boundary condition give the finite volume scheme. Let us define the following discrete H^1_0 semi-norm.

Definition 0.4 *(Discrete energy semi-norm). For any $U^k\in\mathscr{V}^k_h$ (set of constant function on cells), we define*

$$\|U^k\|^2_{1,\Gamma^k_h}\quad:=\quad\sum_{S^k}\sum_{p^k_i\in S^k}\left(U^k_{p_i,j(S)+1/2}-U^k_{p_i,j(S)},U^k_{p_i,j(S)-1/2}-U^k_{p_i,j(S)}\right)Q^k_{sym,p_i,j(S)}\left(U^k_{p_i,j(S)+1/2}-U^k_{p_i,j(S)},U^k_{p_i,j(S)-1/2}-U^k_{p_i,j(S)}\right)^\top \tag{8}$$

where the sub-edge values $U^k_{p_i,j(S)+1/2}$ *are defined by (5), and the boundary values are taken to be uniformly zero.* $Q^k_{sym,p_i,j(S)} = \left(Q^k_{p_i,j(S)} + (Q^k_{p_i,j(S)})^\top\right)/2$ *denotes the symmetric part of* $Q^k_{p_i,j(S)}$ *defined in (6).*

We are now able to establish the main properties of the scheme.

Proposition 0.5 *The problem (7) has a unique solution.*

A PRIORI ESTIMATES

For simplicity in the analysis, we assume X^k_S in the convex hull of the vertices of S^k, as well as the following:

$$|\Upsilon^k(t_{k+1}, X^k_S) - X^{k+1}_S| \leq Ch\tau, \quad |\Upsilon^k(t_{k+1}, X^k_{p_i,j+1/2}) - X^{k+1}_{p_i,j+1/2}| \leq Ch\tau, \quad |\Upsilon^k(t_{k+1}, y^k_{p_i,j+1/2}) - y^{k+1}_{p_i j+1/2}| \leq Ch\tau \tag{9}$$

where $y^k_{p_i,j+1/2}$ is the point that defines the sub-edge $\sigma^k_{p_i,j+1/2}$ together with p_i, $\Upsilon^k(t_{k+1}, X^k_S)$, $\Upsilon^k(t_{k+1}, X^k_{p_i,j+1/2})$ and $\Upsilon^k(t_{k+1}, y^k_{p_i,j+1/2})$ denote the points with the same barycentric coordinate in the convex hull of the image of the vertices of S^k through Φ.

Theorem 0.6 *(Discrete* $\mathbb{L}^\infty(\mathbb{L}^2), \mathbb{L}^2(\mathbb{H}^1)$ *energy estimate). Let* $\{U^k\}_{k=1,\cdots,k_{max}}$ *be the discrete solution of (7) for a given discrete initial data* $U^0 \in \mathscr{V}^0_h$ *and the homogenous boundary condition, then there exists a constant C depending solely on* t_{max} *such that*

$$\max_{k=1,\cdots,k_{max}} \|U^k\|^2_{\mathbb{L}^2(\Gamma^k_h)} + \sum_{k=1}^{k_{max}} \tau \|U^k\|^2_{1,\Gamma^k_h} \leq C\left(\|U^0\|^2_{\mathbb{L}^2(\Gamma^0_h)} + \tau \sum_{k=1}^{k_{max}} \|G^k\|^2_{\mathbb{L}^2(\Gamma^k_h)}\right) \tag{10}$$

Theorem 0.7 *Discrete* $\mathbb{H}^1(\mathbb{L}^2), \mathbb{L}^\infty(\mathbb{H}^1)$ *energy estimate). We assume the sub-matrices* $Q^k_{p_i,j}$ *(cf. (6)) to be symmetric for any sub-cell* $S^k_{p_i,j}$ *around* p^k_i*. Let* $\{U^k\}_{k=1,\cdots,k_{max}}$*, be the discrete solution of (7) for given discrete initial data* $U^0 \in \mathscr{V}^0_h$ *and the homogenous boundary condition, then there exists a constant C depending solely on* t_{max} *such that*

$$\sum_{k=1}^{k_{max}} \tau \|\partial^\tau_t U^k\|^2_{\mathbb{L}^2(\Gamma^k_h)} + \max_{k=1,\cdots,k_{max}} \|U^k\|^2_{1,\Gamma^k_h} \leq C\left(\|U^0\|^2_{\mathbb{L}^2(\Gamma^0_h)} + \|U^0\|^2_{1,\Gamma^0_h} + \tau \sum_{k=1}^{k_{max}} \|G^k\|^2_{\mathbb{L}^2(\Gamma^k_h)}\right) \tag{11}$$

where $\partial^\tau_t U^k = \frac{U^k - U^{k-1}}{\tau}$ *is defined as a difference quotient in time.*

CONVERGENCE

Theorem 0.8 *(Error estimate). We define the piecewise constant error functional on* Γ^k_h *for* $k = 1, \cdots, k_{max}$

$$E^k := \sum_{S^k} \left(u^{-l}(t_k, X^k) - U^k_S\right) \chi_{S^k}$$

measuring the pull back $u^{-l}(t_k, \cdot) = u(t_k, \mathscr{P}^k(\cdot))$ *of the continuous solution* $u(t_k, \cdot)$ *of (1) at time* t_k *and the finite volume solution* U^k *of (7).* χ_{S^k} *denotes the characteristic function of the cell* S^k*. Furthermore, let us assume that* $\| E^0 \|_{\mathbb{L}^2(\Gamma^0_h)} \leq C\,h$*, then the error estimate*

$$\max_{k=1,\cdots,k_{max}} \| E^k \|^2_{\mathbb{L}^2(\Gamma^k_h)} + \tau \sum_{k=1}^{k_{max}} \| E^k \|^2_{1,\Gamma^k_h} \leq C\,(h+\tau)^2 \tag{12}$$

holds for a constant C depending on the regularity assumptions and the time t_{max}*.*

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